Robust Control of Self-Adjoint Distributed-Parameter Structures

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This paper examines the active vibration control of distributed-parameter structures in which a self-adjoint differential operator expresses the stiffness distribution. For large and complex structures, computational requirements and/or modeling limitations ensure that a reduced-order controller is used. However, although in practice only discrete actuators and discrete sensors are available, spatially distributed control forces and spatially distributed observations are desirable for implementing a reduced-order controller. Therefore a distinction arises among 1) designing distributed control forces for a reduced-order model, 2) implementing the control forces with a number of actuators, and 3) estimating the distributed state from a number of sensors. Herein the distinctions are realized by introducing three appropriate projection operations. The effects of the three projections on the actual closed-loop eigenvalues are investigated in detail. A criterion for the controller to be robust in the stability sense is discussed and illustrative examples are presented.

I. Introduction

ACTUAL flexible structures, e.g., large space structures, can have complicated geometry and they possess mass and stiffness properties that are spatially distributed. Recently, the active vibration control of such structures has received a great deal of attention.[1-3] Before a controller can be designed, a mathematical model representing the distributed-parameter structure must be formulated. Althrough the structure's motion is governed by a system of partial differential equations, the structure's complexity, e.g., its composition of many different structural elements, almost always prohibits writing the system of partial differential equations explicitly. Moreover, even if the equations can be written, their closed-form solution is probably impossible to obtain. Therefore, a finite element model of the structure is usually formulated. The finite element model is a spatial discretization of the system of partial differential equations. Presumably, it can be formulated without explicit knowledge of the partial differential equations. The result is a discrete set of ordinary differential equations in time. Of course, a solution of the discrete equations approximates the structure's actual motion. The difference between the approximate motion and the actual motion depends on the finite element model. The error in the approximate motion can be bounded by use of a priori knowledge about the finite element discretization.[4,5]

A major problem of vibration control arises because it is necessary for the controller to be based on a reduced-order mathematical model, i.e., a model with a small number of degrees of freedom. The finite element model is usually not the desired reduced-order model. An inherent property of many finite element models is that they possess a very large number of degrees of freedom. It is common to approximate the structure's modeled motion, i.e., to further approximate the actual motion, by representing the modeled motion as the sum of a small number of eigenvectors of the finite element model, where each eigenvector multiplies a time-dependent generalized coordinated. A reduced-order model in terms of the chosen eigenvectors is formulated and the controller is designed for this reduced-order model. The eigenvectors approximate the structure's actual lower modes of vibration and the approach is called modal control. The eigenvectors that make up the reduced model, i.e., the approximate modes, are called controlled modes. The remaining eigenvectors can be called residual modes. Of course, in addition to the modeled coordinates, an infinity of other coordinates (called residual coordinates) are necessary to represent the actual structure completely. Residual coordinates include all of the residual modes as well as all other unmodeled coordinates.

The difficulties of concern in much of the recent literature[6-21] occur because the controller for the reduced-order model inevitably must be used to control the actual structure. Because implementable control forces must be exerted at discrete points rather than being spatially distributed, the energy used to control the reduced-order model actually will excite the residual coordinates. This problem is referred to as control spillover.[6,7] In addition, position and velocity sensors located at specific points on the structure must be used to estimate the spatially distributed displacement and velocity functions, i.e., the distributed state. The estimated state rather than the actual state is used in the feedback control system, and it usually contains a dependence on the residual coordinates where the dependence is known as observation spillover.[6,7] The use of the estimated state in conjunction with the control spillover and modeling errors alters the closed-loop behavior of the controlled modes. Using the reduced controller also can lead to closed-loop instabilities in the residual coordinates.[6,7]

This paper is concerned with the active vibration control of distributed-parameter structures in which a self-adjoint differential operator expresses the stiffness distribution. For simplicity, the control gains are assumed to be constant, so that the actual closed-loop system does not have time-varying coefficients. The method of controller design is not the main concern and either pole placement or optimal control to minimize a quadratic cost functional over an infinite time interval can be used. It is important to keep in mind that when using discrete actuators and/or sensors in structural vibration control a great deal of freedom exists in determining their locations; however, spatially distributed control forces and spatially distributed observations are desirable although they are not often realizable. In fact, a "full state feedback" controller for the reduced-order model is in terms of

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distributed forces, where the space of distributed forces is spanned by the functions used to form the reduced-order model. Therefore, one must eventually distinguish among 1) designing distributed control forces, 2) implementing the control forces with a number of actuators, and 3) estimating the spatially distributed state from a number of sensors. The distinctions here are additionally motivated by the methodology developed in Refs. 11-17, particularly the developments of Ref. 15, for independently controlling a number of lower modes, where the control method is termed Independent Modal Space Control (IMSC). For other control methods, these three distinctions may not be as readily apparent.

Section II considers the distributed-parameter structure and its control by a finite dimensional controller. Three mathematical projections are introduced: a projection onto the reduced-order structural model, a projection for approtioning the control forces, and a projection for estimating the distributed state. The idea of a state estimation projection operator is based on the idea of "modal filters" developed in Ref. 15. Also, in Sec. II a criterion for the closed-loop distributed system to be robust in the stability sense is discussed in terms of the three projection operators. It is shown that the closed-loop system is stable when the projection operator for the control is the adjoint of the projection operator for estimating the state. This represents a generalization of the well-known robustness resulting from collocation of discrete actuators and discrete sensors. Section III discusses how the three projection operations affect the closed-loop system's eigenvalues. The results constitute an application to the control problem of the spectral approximation theory found in Ref. 22. Several general conclusions are permitted concerning the accuracy of the reduced-order model, the accuracy of the control projection operator, the accuracy of the state estimation projection operator, and the closeness of the desired closed-loop eigenvalues to the actual closed-loop eigenvalues. Next, the independent modal space control method[11-17] is considered in Sec. IV. The method is computationally advantageous and conveniently illustrates the theory of Secs. II and III. Finally, Sec. V presents numerical examples that illustrate the closed-loop errors resulting from using reduced-order controllers.

II. Control of Distributed-Parameter Structures

First, the partial differential equations governing the structure's motion are considered. Although the actual equations usually cannot be written, many properties of the equations are known. For simplicity, the structure's motion is assumed to be governed by the *single* equation

$$\rho(x)\ddot{u}(x,t)+c_1\dot{u}(x,t)+Lu(x,t)=f(x,t), \quad x\epsilon\Omega, \quad t\geq 0 \qquad (1)$$

which must be satisfied at every point x of the domain Ω of the structure and at every time $t\geq 0$. A *system* of equations for a complicated structure also could be written symbolically in the form of Eq. (1). In Eq. (1), a dot denotes a partial derivative with respect to time, $u(x,t)$ the displacement of a point x at time t, $\rho(x)$ the mass distribution, c_1 a self-adjoint nonnegative linear differential operator of even order $\leq 2p$, L a self-adjoint coercive linear differential operator of order $2p$ $(p\geq 1)$ with a compact inverse L^{-1}, and $f(x,t)$ a control force exerted at point x. In addition to Eq. (1), $u(x,t)$ must satisfy the following boundary conditions and initial conditions, respectively,

$$B_i[u(x,t)]=0, \quad i=0,1,...,p-1, \quad x\epsilon\delta\Omega; \quad t\geq 0 \qquad (2)$$

$$u(x,0)=u_0(x), \quad \dot{u}(x,0)=u_1(x), \quad x\epsilon\Omega+\delta\Omega \qquad (3)$$

where $\delta\Omega$ is the smooth boundary of Ω. Each B_i $(i=0,...,p-1)$ in Eq. (2) is a linear differential operator containing derivatives with respect to the outward normal to $\delta\Omega$ of order q_i, where $0\leq q_i\leq 2p-1$. The boundary conditions (2) are assumed to form a normal covering[4] of L on $\delta\Omega$ and they can represent essential boundary conditions as well as natural boundary conditions. It is recognized that $u(x,t)$, at the least, is in a real Hilbert space H with inner product $(u_1,u_2)=\int u_1u_2 d\Omega$. Moreover, the domain of L, denoted $D(L)$, is dense in another Hilbert space V which is dense in H and the space H is dense in the dual space of V, denoted V'. The space V is taken to be the domain of $L^{1/2}$ with inner product $(v_1,v_2)_V=(L^{1/2}v_1, L^{1/2}v_2)$. The control forces $f(x,t)$ can be taken as elements of H. When a suitable variational formulation is defined and used to replace the differential equation (1), the control forces also can be taken as elements of V'.

Next, if Eqs. (1-3) were known and if arbitrary spatially distributed control forces were available, the *desired* control force $f(x,t)\epsilon H$ would be

$$f(x,t)=-g_0u(x,t)-g_1\dot{u}(x,t) \qquad (4)$$

where g_0 and g_1 are self-adjoint nonnegative linear operators. The method of obtaining g_0 and g_1 is neglected here although the elegant method of Ref. 15 will be considered in Sec. IV. Substituting Eq. (4) into Eq. (1), the closed-loop motion is governed by the equation

$$\rho\ddot{u}+(c_1+g_1)\dot{u}+(L+g_0)u=0 \qquad x\epsilon\Omega; \qquad t\geq 0 \qquad (5)$$

The energy in the closed-loop system is given by

$$E(u,\dot{u})=\tfrac{1}{2}(\dot{u},\rho\dot{u})+\tfrac{1}{2}[u,u] \qquad (6)$$

where $[u,u]=(L^{1/2}u, L^{1/2}u)+(u,g_0u)$. The energy E can be differentiated with respect to time, yielding

$$\dot{E}(u,\dot{u})=(\dot{u},\rho\ddot{u})+[u,\dot{u}]=-(\dot{u},c_1\dot{u}+g_1\dot{u}) \qquad (7)$$

Since c_1 and g_1 are self-adjoint nonnegative operators, $\dot{E}\leq 0$, and the closed-loop system is dissipative. If c_1+g_1 is also a positive definite operator, the closed-loop system is asymptotically stable. The sequel considers that c_1 describes some pervasive damping, however slight, so that the open-loop system, and hence the closed-loop system with g_0 and g_1 as above, is asymptotically stable.

Attention is now turned to a finite element discretization of Eq. (1). To this end, a variational formulation of the boundary initial-value problem descibed by Eqs. (1-3) must be considered first. Multiplying Eq. (1) by a function $v(x)$ and integrating the result over Ω, the variational formulation seeks to find a function $u(x,t)\epsilon V_0$ that satisfies the initial conditions (3) and, at every time $t\geq 0$, satisfies the equation

$$(v,\rho\ddot{u})+(v,c_1\dot{u})+(L^{1/2}v,L^{1/2}u)=(v,f) \qquad (8)$$

for all functions $v(x)\epsilon V$. The Hilbert space V_0 is the subspace of V of functions satisfying the essential boundary conditions contained in conditions (2). Note that the natural boundary conditions have been already invoked in the process of obtaining the inner product $(L^{1/2}v, L^{1/2}u)$ from (v,Lu). The finite element method consists of replacing the above problem with that of finding a function $u(x,t)\epsilon V_h$ such that at every time $t\geq 0$. Eq. (8) is satisfied for all functions $V\epsilon V_h$, where V_h is a finite dimensional subspace of V_0. The subspace V_h is spanned by a set of trial functions $\gamma_i(x)$ $(i=1,2,...,\bar{n})$ that have local support, where $\bar{n}$ is the dimension of V_h (Refs. 4 and 5). In terms of the functions $\gamma_i(x)$, values of the coefficients $\bar{a}_i(t)$ are sought so that

$$u(x,t)=\sum_{i=1}^{\bar{n}}\gamma_i(x)\bar{a}_i(t)=\gamma^T(x)\bar{a}(t)$$

satisfies Eq. (8) for all functions $v \in V_h$. The coefficients $\bar{a}(t)$ are determined by solving the $\bar{n}$ simultaneous ordinary differential equations

$$\bar{M}\ddot{\bar{a}}(t) + \bar{C}\dot{\bar{a}}(t) + \bar{K}\bar{a}(t) = \bar{F}(t) \tag{9}$$

where $\bar{M} = \int \rho\gamma\gamma^T d\Omega$ is the mass matrix, $\bar{C} = \int \gamma c_I \gamma^T d\Omega$ is the damping matrix, $\bar{K} = \int (L^{1/2}\gamma)(L^{1/2}\gamma^T) d\Omega$ is the stiffness matrix, and $\bar{F} = \int \gamma f d\Omega$ is the discretized force vector. The $\bar{n} \times \bar{n}$ matrices $\bar{M}$, $\bar{C}$, and $\bar{K}$ are symmetric and positive definite.

As mentioned in Sec. I, if $\bar{n}$ is a large number, Eq. (9) is usually not a satisfactory reduced-order model of the structure. The order of the model can be reduced further by restricting the search for functions $u(x,t)$ to a subspace of V_h denoted by V_m, where the dimension of V_m is n $(n \leq \bar{n})$. As a basis for V_m, functions $\gamma^T(x)\phi_i$ $(i = 1,2,...,n)$ that have global support can be chosen. The vectors ϕ_i are commonly chosen as eigenvectors of the undamped, uncontrolled finite element model, i.e, they satisfy the eigenvalue problem $\lambda_i^2 \bar{M}\phi_i + \bar{K}\phi_i = 0$. Values of the coefficients $a_i(t)$ are now sought so that

$$u(x,t) = \sum_{i=1}^{n} \gamma^T(x)\phi_i a_i(t) = \gamma^T \Phi_m a(t)$$

satisfies Eq. (8) for all functions $v \in V_m$. The coefficients $a(t)$ are determined by solving the n equations

$$M\ddot{a}(t) + C\dot{a}(t) + Ka(t) = F(t) \tag{10}$$

where $M = \Phi_m^T \bar{M} \Phi_m$, $C = \Phi_m^T \bar{C}_m \Phi_m$, $K = \Phi_m^T \bar{K} \Phi_m$, and $F = \Phi_m^T \bar{F}$. Note that when the columns of Φ_m are eigenvectors, M and K will be diagonal matrices.

Based on Eq. (10), a control force vector F can be obtained that imparts the desired closed-loop characteristics to the reduced-order model. In terms of the reduced coordinates a, F has the form

$$F(t) = -G_0 a(t) - G_I \dot{a}(t) \tag{11}$$

where G_0 and G_I are $n \times n$ symmetric nonnegative-definite matrices. The interest is in using the control described by Eq. (11) to control the actual structure. To this end, it is necessary to use the components of F to obtain an associated control force $f(x,t)$. It is natural to consider the forces $f(x,t)$ to be in V_m. The coefficents vectors $a(t)$ and $\dot{a}(t)$ are the result of projecting the actual displacement $u(x,t)$ and the actual velocity $\dot{u}(x,t)$, respectively, onto the subspace V_m. Therefore, the *natural extension to* V of the control described by Eq. (11) is considered to be

$$f_m(x,t) = -g_{0m} u(x,t) - g_{Im} \dot{u}(x,t) \tag{12}$$

where the nonnegative self-adjoint operators g_{0m} and g_{Im} are given by

$$g_{0m} u = \rho\gamma^T \Phi_m M^{-1} G_0 M^{-1} \Phi_m^T(\gamma, \rho u) \tag{13a}$$

$$g_{Im} \dot{u} = \rho\gamma^T \Phi_m M^{-1} G_I M^{-1} \Phi_m^T(\gamma, \rho \dot{u}) \tag{13b}$$

Because the force $f_m(x,t)$ is distributed spatially, it is in V_m, it can be compared with the desired force $f(x,t)$ of Eq. (4). Denoting by π_m and π_m^* the projection and its adjoint, respectively, from V onto V_m, it would be nice if $g_{0m} = \pi_m^* g_0 \pi_m$ and $g_{Im} = \pi_m^* g_I \pi_m$. In general, this is not true although one can write

$$g_{0m} = \pi_m^* g_0 \pi_m + \hat{g}_0 \tag{14a}$$

$$g_{Im} = \pi_m^* g_I \pi_m + \hat{g}_I \tag{14b}$$

where $\hat{g}_0$ and $\hat{g}_I$ are self-adjoint operators representing small differences in the control law.

It is informative to be more specific about projection operators. In general, the projection πv of a function $v(x)$ is realized by approximating $v(x)$ as the sum $w^T(x)a$, where a is a vector of coefficients and $w(x)$ a vector of functions defined on Ω. The coefficient vector a can be determined so that

$$\pi v = w^T(z, w^T)^{-1}(z, v) \tag{15}$$

where $z(x)$ is also a vector of functions defined on Ω. Obviously, $\pi(\pi v) = \pi v$, i.e., $\pi^2 = \pi$. The adjoint of π, denoted by π^*, also has the form of Eq. (15) except that the roles of z and w are interchanged. In practice, it is nice if z is orthonormal to w, i.e., $(z, w^T) = I$, although orthonormality is not essential. A projection in which z and w are orthogonal is called an orthogonal projection. Note that Eqs. (13) imply π_m in the form of Eq. (15) with $z = \rho\Phi_m^T\gamma$ and $w = \Phi_m^T\gamma$, and π_m^* with $z = \Phi_m^T\gamma$, and $w = \rho\Phi_m^T\gamma$. The distinction between π_m and π_m^* is the result of considering the unweighted inner product $(u_I, u_2) = \int u_I u_2 d\Omega$ rather than an inner product weighted by the mass distribution $\rho(x)$. Recall that γ is a vector of finite element trial functions having local support.

Although the natural extension to V of the control for the reduced model is given by Eq. (12), one is not often able to implement the control as given because of restrictions on the actuating and sensing devices available. Let us consider that n_c actuators are available to apply a control force in another space V_c which is n_c dimensional. Moreover, let us consider that n_s sensors are available to measure that part of u and $\dot{u}$ in a space V_s which is n_s dimensional. Both V_c and V_s are subspaces of V' although they need not be subspaces of V_m or vice versa. We can denote by π_c and π_s projections from V onto V_c and V_s, respectively, and consider implementing the reduced control via these projections. The resulting force $f_a(x,t)$ is the force actually applied to control the structure and it can be written symbolically as

$$f_a(x,t) = -g_{0a} u(x,t) - g_{Ia} \dot{u}(x,t) \tag{16}$$

where

$$g_{0a} = \pi_c g_{0m} \pi_s = \pi_c \pi_m^* g_0 \pi_m \pi_s + \pi_c \hat{g}_0 \pi_s \tag{17a}$$

$$g_{Ia} = \pi_c g_{Im} \pi_s = \pi_c \pi_m^* g_I \pi_m \pi_s + \pi_c \hat{g}_I \pi_s \tag{17b}$$

Note that the variational formulation permits $f_a \in V'$, i.e., the control forces can be distributions such as Dirac delta functions.

The control force $f_a(x,t)$ replaces $f(x,t)$ in Eq. (1). The closed-loop system with f_a acting in place of f cannot be assumed to be asymptotically stable. Control spillover and observation spillover are embodied in the projections π_c and π_s, respectively. The presence of control spillover, observation spillover, and modeling errors can lead to closed-loop instabilities.[6,7] Nevertheless, as in the earlier discussion, all instabilites can be avoided if g_{Ia} and g_{0a} are self-adjoint nonnegative operators. This suggests that it is desirable for $\pi_c = \pi_s^*$. Note that g_{0m} and g_{Im} are already nonnegative and self-adjoint. Thus, when $\pi_c = \pi_s^*$, the closed-loop system will always be asymptotically stable regardless of the magnitude of modeling errors and deviations of the actual control f_a from the desired control f. Hence, $\pi_c = \pi_s^*$ constitutes a simple robustness criterion. The critrion generalizes the robustness criterion for collocated discrete actuators and sensors discussed in Refs. 8 and 19-21.

The question remains as to how well the control force f_a performs, regardless of whether $\pi_c = \pi_s^*$, as a replacement for f. It is possible that the internal damping represented by the operator c_I is sufficient for the closed-loop system to be stable for an appropriate control force f_a with $\pi_c \neq \pi_s^*$. On the other hand, for $\pi_c = \pi_s^*$ and guaranteed stability, f_a may not control the structure satisfactorily. The effects of π_c, $\pi_m\,(\pi^*_m)$, and π_s on the eigenvalues of the actual closed-loop system are examined in the next section.

III. Eigenvalues of the Actual Closed-Loop System

One means of comparing the performance of the actual control force f_a, given by Eq. (16), to that of the desired cotnrol force f, given by Eq. (4), is to compare the eigenvalues of the two resulting closed-loop systems. Two corresponding eigenvalue problems must be formulated first. It will prove convenient to define the auxiliary coordinate $v(x,t) = \dot{u}(x,t)$ and rewrite Eq. (1) in the form

$$\rho(x)\dot{y}(x,t) = \bar{A}y(x,t) + b(x,t), \qquad x\in\Omega; \quad t\geq 0 \tag{18}$$

where $y(x,t) = \{u(x,t),\;\; v(x,t)\}^T$ and

$$\bar{A} = \begin{bmatrix} 0 & \rho(x) \\ -L & -c_I \end{bmatrix}, \; b(x,t) = \begin{Bmatrix} 0 \\ f(x,t) \end{Bmatrix} \tag{19}$$

Then, after replacing f with its explicit form [Eq. (4)] and substituting $y(x,t) = e^{\lambda t}y(x)$ into Eq. (18) to eliminate the time dependence, the eigenvalue problem of the *desired closed-loop system* is obtained in the form

$$\lambda\rho(x)y(x) = Ay(x), \qquad x\in\Omega \tag{20}$$

where

$$A = \bar{A} + A_c \tag{21a}$$

$$A_c = \begin{bmatrix} 0 & 0 \\ -g_0 & -g_I \end{bmatrix} \tag{21b}$$

In Eqs. (21), $\bar{A}$ is a linear differential operator representing the uncontrolled structure and A_c is a linear operator specifying the desired feedback control. Of course, after substituting the explicit form (16) of f_a for f in Eq. (18), the eigenvalue problem of the *actual closed-loop system* is obtained in the form

$$\lambda_a\rho(x)y_a(x) = A_a y_a(x), \qquad x\in\Omega \tag{22}$$

where $A_a = \bar{A} + \pi_c\pi_m^* A_c \pi_m \pi_s + \pi_c \hat{A}_c \pi_s$ is defined using the notation

$$\pi_c\pi_m^* A_c \pi_m \pi_s = \begin{bmatrix} 0 & 0 \\ -\pi_c\pi_m^* g_0 \pi_m \pi_s & -\pi_c\pi_m^* g_I \pi_m \pi_s \end{bmatrix} \tag{23a}$$

$$\pi_c \hat{A}_c \pi_s = \begin{bmatrix} 0 & 0 \\ -\pi_c \hat{g}_0 \pi_s & -\pi_c \hat{g}_I \pi_s \end{bmatrix} \tag{23b}$$

In the following discussion, the idea that A_a is an approximation of A is exploited. As in Ref. 22, A_a is assumed to converge pointwise to A as an intrinsic parameter approaches infinity. Herein, the convergence of A_a to A depends on $\pi_m\,(\pi_m^*)$, π_s, and π_c, and, therefore, on n, n_s, and n_c. Because π_s and π_c must converge to identify operators, V_c and V_s must be considered here as subspaces of V rather than V', e.g., control forces in the form of distributions are excluded in this section although V_c can be spanned by functions in V with local support that closely approximate distributions. Rather than explicitly identifying a specific convergence parameter, it is sufficient to assume that A_a is close enough to A so that the ideas reviewed in Ref. 22 are applicable. The point is to obtain some understanding with a minimum of mathematical details.

Equation (20), in conjunction with the boundary conditions (2), defines an eigenvalue problem for each desired isolated eigenvalue λ of finite algebraic multiplicity m. Note that A and A_a operate on functions $y(x)$ in a complex Hilbert space Y, and let $\mathrm{M} = PY$ ($\mathrm{M}^* = P^*Y$) denote the invariant subspace (adjoint subspace) associated with λ, where $P(P^*)$ is the spectral projection (adjoint projection). Of course, Eq. (22), in conjunction with the boundary conditions (2), defines an eigenvalue problem for each actual eigenvalue λ_a. Let λ_{aj} be the distinct actual eigenvalues inside a closed Jordan curve Γ which isolates a desired eigenvalue λ. As in Ref. 22, if P_{aj} is λ_{aj}'s spectral projection, $P_a = \Sigma_j P_{aj}$, and then $\mathrm{M}_a = P_a Y$ is the invariant subspace associated with all the actual eigenvalues inside Γ. It is shown in Ref. 22 that the dimension of $\mathrm{M_a}$ equals the dimension of M provided A^{-1} and A_a^{-1} are compact and A_a is close enough to A. A precise definition of closeness of A_a to A is not required in the following discussion and is omitted. The spectral projection P_a also converges to P in an appropriate sense (the precise sense of convergence is omitted) and the intersection of the spectrum of Eq. (22) with the interior of Γ consists of exactly m eigenvalues λ_{aj} $(j=1,2,\ldots,m)$. Therefore, one can approximate λ by the arithmetic mean

$$\lambda_a = \frac{1}{m}\sum_{j=1}^{m}\lambda_{aj}$$

and consider error bounds for $\lambda - \lambda_a$.

Note that each λ can be obtained as a stationary value of the generalized Rayleigh quotient

$$\lambda = R(\bar{y},\bar{y}^*) = (\bar{y}^*, A\bar{y})/(\bar{y}^*,\rho\bar{y}) \qquad \bar{y},\bar{y}^*\in Y \tag{24}$$

where, using the notation $\bar{y} = \{y_I, y_2\}^T$ and $\bar{y}^* = \{y_I^*, y_2^*\}^T$, the inner products in Eq. (24) are $(\bar{y},\rho y) = (y_I^*\rho y_I) + (y_2^*,\rho y_2)$ and $(\bar{y}^*, A\bar{y}) = (y_I^*,\rho y_2) - (L^{1/2}y_2^*, L^{1/2}y_I) - (y_2^*, g_0 y_I) - (y_2^*, c_I y_2 + g_I y_2)$. The stationarity conditions for $R(\bar{y},\bar{y}^*)$ are precisely the eigenvalue problem (20) and its adjoint. Hence, an eigenvalue λ of algebraic multiplicity m can be written in terms of $y_i\in\mathrm{M}$, $y_i^*\in\mathrm{M}^*$, and $\bar{y},\bar{y}^*\in Y$ as

$$\lambda = \frac{1}{m}\sum_{i=1}^{m}(y_i^*, A\bar{y})/(y_i^*,\rho\bar{y}) = \frac{1}{m}\sum_{i=1}^{m}(\bar{y}^*, Ay_i)/(\bar{y}^*,\rho y_i) \tag{25}$$

where $\bar{y}$ and $\bar{y}^*$ cannot be an eigenfuction and an adjoint eigenfunction, respectively, but they are otherwise arbitrary functions in Y. Of course, each λ_{aj} can be obtained as a stationary value of a generalized Rayleigh quotient similar to Eq. (24) but with A replaced by A_a. Therefore, for y_{ai} (y_{ai}^*) $(i=1,2,\ldots,m)$ a basis (adjoint basis) for $\mathrm{M}_a(\mathrm{M}_a^*)$, $Py_{ai}\in\mathrm{M}$, $P^*y_{ai}^*\in\mathrm{M}$, for y_i and y_i^* $(i=1,2,\ldots,m)$ a basis (adjoint basis) for M (M*), $P_a y_i \in \mathrm{M}_a$, $P_a^* y_i^* \in \mathrm{M}_a^*$, and A_a close enough to A, one obtains

$$\begin{aligned}\lambda - \lambda_a &= \frac{1}{m}\sum_{i=1}^{m}(y_{ai}^*, (A-A_a)Py_{ai})/(y_{ai}^*,\rho Py_{ai}) \\ &= \frac{1}{m}\sum_{i=1}^{m}(P^*y_{ai}^*, (A-A_a)y_{ai})/(P^*y_{ai}^*,\rho y_{ai}) \\ &= \frac{1}{m}\sum_{i=1}^{m}(y_i^*, (A-A_a)P_a y_i)/(y_i^*,\rho P_a y_i) \\ &= \frac{1}{m}\sum_{i=1}^{m}(P_a^* y_i^*, (A-A_a)y_i)/(P_a^* y_i^* \rho y_i) = O(\alpha)\end{aligned} \tag{26}$$

where $\alpha = \min(\|(A-A_a)P\|, \|(A^*-A_a^*)P^*\|)$, i.e., $|\lambda - \lambda_a| \leq d_I\alpha$. In this discussion, the first two equalities in Eq. (26) are used although one could also consider the second two equalities and operator norms in terms of P_a and P_a^*.

Equation (26) shows the first-order accuracy of λ_a as an approximation of each desired eigenvalue λ. In addition, Ref. 22 shows that the gap between the two subspaces M and M_a is also of order α, i.e., $O(\alpha)$.

It is desirable to understand how α depends on π_m, π_m^*, π_c, and π_s. To this end, write

$$A - A_a = (1 - \pi_c \pi_m^*) A_c \pi_m \pi_s + A_c (1 - \pi_m \pi_s) - \pi_c \hat{A}_c \pi_s \tag{27a}$$

$$A - A_a = \pi_c \pi_m^* A_c (1 - \pi_m \pi_s) + (1 - \pi_c \pi_m^*) A_c - \pi_c \hat{A}_c \pi_s \tag{27b}$$

Then, defining

$$\epsilon_1 = \| (1 - \pi_c) \pi_m^* A_c \pi_m \pi_s P \|$$

$$\epsilon_2 = \| (1 - \pi_m^*) A_c \pi_m \pi_s P \|$$

$$\epsilon_3 = \| A_c \pi_m (1 - \pi_s) P \|$$

$$\epsilon_4 = \| A_c (1 - \pi_m) P \|$$

$$\epsilon_5 = \| \pi_c \hat{A}_c \pi_s P \|$$

$$\epsilon_1^* = \| (1 - \pi_s^*) \pi_m^* A_c^* \pi_m \pi_c^* P^* \|$$

$$\epsilon_2^* = \| (1 - \pi_m^*) A_c^* \pi_m \pi_c^* P^* \|$$

$$\epsilon_3^* = \| A_c^* \pi_m (1 - \pi_c^*) P^* \|$$

$$\epsilon_4^* = \| A_c^* (1 - \pi_m) P^* \|$$

$$\epsilon_5^* = \| \pi_s^* \hat{A}_c^* \pi_c^* P^* \|$$

it follows that

$$\alpha = \min[\beta, \beta^*] = \min[\max(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5), \max(\epsilon_1^*, \epsilon_2^*, \epsilon_3^*, \epsilon_4^*, \epsilon_5^*)] \tag{28}$$

Next, let us search for a more accurate bound on $|\lambda - \lambda_a|$. If $P_{(a)}$ denotes the restriction of P to M_a, then $P_{(a)}^{-1}$ maps elements of M into M_a. Hence, $P_a y_i$ and $P_{(a)}^{-1} y_i$ are colinear and P_a can be replaced with $P_{(a)}^{-1}$ in Eq. (26). Upon substituting Eq. (27a) into the third form of Eq. (26) with P_a replaced by $P_{(a)}^{-1}$ and performing some manipulations, one arrives finally at

$$\begin{aligned}
\lambda - \lambda_a = \frac{1}{m} \sum_{i=1}^{m} \Big\{ \Big[& ((1 - \pi_m \pi_c^*) \overset{*}{P} y_i^*, A_c \pi_m \pi_s (P_{(a)}^{-1} P - P) y_i) \\
& - ((1 - \pi_m \pi_c^*) P^* y_i^*, A_c (1 - \pi_m \pi_s) P y_i) \\
& + ((1 - \pi_m \pi_c^*) \overset{*}{P} y_i^*, (1 - \pi_c \pi_m^*) A_c P y_i) \\
& + ((1 - \pi_s^* \pi_m^*) A_c P^* y_i^*, (P_{(a)}^{-1} P - P) y_i) \\
& + ((1 - \pi_s^* \pi_m^*) A_c^* \overset{*}{P} y_i^*, (1 - \pi_m \pi_s) P y_i) \\
& - (y_i^*, \pi_c \hat{A}_c \pi_s P_{(a)}^{-1} y_i) \Big] \Big/ (y_i^*, \rho P_{(a)}^{-1} y_i) \Big\}
\end{aligned} \tag{29}$$

By using $\| P_{(a)}^{-1} P - P \| \leq d_3 \alpha$, Eq. (29) permits a higher accuracy bound on $\lambda - \lambda_a$ when $A_c P = P A_c$ and $A_c^* P^* = P^* A_c^*$. Note that an analogous equation can be obtained after substituting Eq. (27b) into the fourth form of Eq. (26) with P_a replaced by $P_{(a)}^{-1}$. It is easily seen that $A_c P = P A_c$ and $A_c^* P^* = P^* A_c^*$ when eigenfunctions and adjoint eigenfuctions of the uncontrolled structure are also eigenfunctions and adjoint eigenfunctions of the desired controlled structure, i.e., when $\bar{A} P = P \bar{A}$ it is easy to conclude that $(\bar{A} + A_c) P = P(\bar{A} + A_c)$ implies $A_c P = P A_c$. Defining

$$\epsilon_6 = \| \pi_m \pi_s (P_{(a)}^{-1} P - P) \|, \qquad \epsilon_7 = \| P_{(a)}^{-1} P - P \|$$

$$\epsilon_c = \| (1 - \pi_c) \pi_m^* P \|, \qquad \epsilon_c^* = \| \pi_m (1 - \pi_c^*) P^* \|$$

$$\epsilon_s = \| \pi_m (1 - \pi_s) P \|, \qquad \epsilon_s^* = \| (1 - \pi_s^*) \pi_m^* P^* \|$$

$$\epsilon_m = \| (1 - \pi_m) P \|, \qquad \epsilon_{m^*} = \| (1 - \pi_m) P^* \|,$$

$$\epsilon_m^* = \| (1 - \pi_m^*) P \| \qquad \epsilon_{m^*}^* = \| (1 - \pi_m^*) P^* \|,$$

$$\hat{\epsilon} = \epsilon_5^* / \| \hat{A}_c \|$$

one finds that

$$\begin{aligned}
\lambda - \lambda_a = O[\max(& \epsilon_c^* \epsilon_6, \epsilon_{m^*} \epsilon_6, \epsilon_s^* \epsilon_7, \epsilon_{m^*}^* \epsilon_7, \epsilon_c^* \epsilon_c, \epsilon_c^* \epsilon_s, \\
& \epsilon_c^* \epsilon_m, \epsilon_c^* \epsilon_m^*, \epsilon_s^* \epsilon_s, \epsilon_s^* \epsilon_m, \epsilon_m^* \epsilon_c, \epsilon_{m^*} \epsilon_s, \epsilon_{m^*} \epsilon_m, \epsilon_{m^*} \epsilon_m^*, \\
& \epsilon_{m^*}^* \epsilon_s, \epsilon_{m^*}^* \epsilon_m, \hat{\epsilon})]
\end{aligned} \tag{30}$$

Note that generally $O(\epsilon_m) = O(\epsilon_{m^*}) = O(\epsilon_m^*) = O(\epsilon_{m^*}^*)$.

The possible double-order accuracy is a strong argument in favor of a desired controller that preserves the uncontrolled eigenfunctions. This is the idea of independent modal space control developed in Refs. 11-17. However, note that the desired control must theoretically take into account the structure's internal damping in order to obtain the higher accuracy. In practice, the internal damping is small and difficult to model. Hence, it is usually neglected in designing the reduced-order control and attaining the higher accuracy depends on the validity of neglecting the internal damping.

Equation (29), i.e., the possible double-order accuracy, also provides a strong argument for only controlling a number of lower modes of the structure, i.e., for modal control. By considering a desired controller that only affects the controlled modes, $A_c P = A_c^* P^* = 0$ when P, P^* are spectral projections associated with the uncontrolled modes. Note that here the concern is with modes which are *theoretically desired to be uncontrolled.* Therefore,

$$(P^* y_{ai}^*, (A - A_a) y_{ai}) = (\pi_s^* \pi_m^* A_c^* (1 - \pi_m \pi_c^*) P^* y_{ai}^*, y_{ai}) + (P^* y_{ai}^*, \pi_c \hat{A}_c \pi_s y_{ai})$$

and

$$(y_{ai}^*, (A - A_a) P y_{ai}) = (y_{ai}^*, \pi_c \pi_m^* A_c (1 - \pi_m \pi_s) P y_{ai}) + (y_{ai}^*, \pi_c \hat{A}_c \pi_s P y_{ai})$$

so that for the uncontrolled modes $\alpha = O$ min $[\max(\epsilon_{m^*}, \epsilon_c^*, \hat{\epsilon}), \max(\epsilon_m, \epsilon_s, \hat{\epsilon})]\}$. This result is well known.[10,15,16] Assuming that $\hat{\epsilon} = O(\epsilon_m)$, it shows how the accuracy of the eigenvalues in the presence of control and observation spillover as well as modeling errors depends on π_m, π_s, and π_c. Indeed, for a given π_c (alternately π_s), the eigenvalues associated with an uncontrolled mode can be moved closer to the open-loop eigenvalues by improving the worst of π_m and π_s (alternately π_c). Moreover, for a given π_c and π_s, a structural model need only be as good as the best of π_c and π_s. *This suggests that a highly accurate structural model is a waste of effort unless it is accompanied by highly accurate sensing or actuating.* Finally, α as well as the higher accuracy estimates must be determined separately for each uncontrolled mode. In fact, the order of accuracy of the uncontrolled modes is erractic and can vary dramatically.

It is important to observe that the theoretical order of accuracy is only of an eigenvalue's modulus. Although the moduli may be highly accurate, their real parts can be positive. For the infinity of residual coordinates of the actual closed-loop system to be asymptotically stable, it is sufficient to require that $\pi_c = \pi_s^*$.

The above discussion tacitly assumes that the *actual control force* f_a converges to the *desired control force* f. For convergence to occur, the method for producing the controller for the reduced-order model must be inherently the same as a method for producing the desired controller. By separating the controller design from its implementation with actuators and sensors, methods such as pole placement and modal space control are permitted. However, if a large number of modes are controlled with only a few discrete actuators and discrete sensors, in the actual closed-loop system one should not expect the eigenvalues of higher modulus, i.e., those associated with the higher controlled modes, to be close to the desired closed-loop eigenvalues.

Finally, note that this discussion is purely theoretical because the partial differential equation of the structure is not available explicitly. Only the finite element model of dimension $\bar{n}$ is available. It is natural to examine the eigenvalues of the closed-loop system consisting of the actual control f_a projected onto V_h and applied to the finite element model. The finite element model then constitutes a test model and bounds on the deviation of the test model's eigenvalues from the actual closed-loop system's eigenvalues can be produced in the same manner considered above. Of course, a more accurate test model consisting of a finite element model of dimension greater than $\bar{n}$ can be considered also. Finding the eigenvalues of a test model is, of course, unnecessary if a priori knowledge of ϵ_m, ϵ_{m^*}, $\hat{\epsilon}$, ϵ_s, and ϵ_c is available and/or $\pi_c = \pi_s^*$. This paper, however, does not seek quantitative estimates of the parameters ϵ.

IV. Independent Modal Space Control and Its Implementation

It is illustrative to use the independent modal space control method developed in Refs. 11-17: 1) to determine a desired controller for the structure, and 2) to determine a reduced-order controller for the structure.

In independent modal space control for the distributed-parameter structure, the eigenfunctions $\psi_i(x)$ of the structure are used to replace the partial differential equation (1) by an infinite number of uncoupled simultaneous equations. For example, assuming that c_1 is extremely small and can be safely neglected, one can write

$$u(x,t) = \sum_{i=1}^{\infty} \psi_i(x) u_i(t),$$

substitute the sum into Eq. (1), and obtain the equations

$$\ddot{u}_i + \omega_i^2 u_i = f_i, \qquad i = 1,2,\ldots,\infty; \quad t \geq 0 \tag{31}$$

where $\psi_i(x)$ is an orthonormal eigenfunction of the undamped, uncontrolled structure, i.e., $\omega_i^2 \rho \psi_i = L\psi_i$, $(\psi_i, L\psi_j) = \omega_i^2 \delta_{ij}$ and $f_i = (\psi_i, f)$. Then, determining a controller for each mode i in a set of n critical modes ($i = 1,\ldots,n$) is equivalent to determining the coefficients g_{0i} and g_{1i} in $f_i(t) = -g_{0i} u_i(t) - g_{1i} \dot{u}_i(t)$. The coefficients g_{0i} and g_{1i} can be chosen to make the closed-loop mode have a desired eigenvalue or they can be obtained as the solution of a 2×2 matrix Ricatti equations for each mode i.[15]. Of course, the control forces f yielding the proper modal force f_i are spatially distributed. Moreover, each modal force f_i depends on obtaining $u_i(t)$ and $\dot{u}_i(t)$ from $u(x,t)$ and $\dot{u}(x,t)$ via the modal filters $u_i = (\psi_i, \rho u)$ and $\dot{u}_i = (\psi_i, \dot{\rho} u)$.[15] Therefore, the desired control force $f(x,t)$ is given by Eq. (4) with

$$g_0 u = \sum_{i=1}^{n} \rho(x) \psi_i(x) g_{0i}(n_i, \rho u) \tag{32a}$$

$$g_1 \dot{u} = \sum_{i=1}^{n} \rho(x) \psi_i(x) g_{1i}(\psi_i, \rho \dot{u}) \tag{32b}$$

An analogous procedure can be used to determine an approximate control force $f_m(x,t)$. One can write

$$\bar{a}(t) = \sum_{i=1}^{n} \phi_i \bar{u}_i$$

where ϕ_i are orthonormal eigenvectors satisfying $\bar{\omega}_i^2 \bar{M} \phi_i = \bar{K} \phi_i$, substitute the sum into Eq. (9), and obtain the equations $\ddot{\bar{u}}_i + \bar{\omega}_i^2 \bar{u}_i = \bar{f}_i$ $(i = 1,2,\ldots,n)$ where $\phi_i^T \bar{K} \phi_j = \bar{\omega}_i^2 \delta_{ij}$ and $\bar{f}_i = \phi_i^T \bar{F}$. Then, for each approximate mode i $(i = 1,2,\ldots,n)$ the coefficients $\bar{g}_{0i}$ and $\bar{g}_{1i}$ in $\bar{f}_i = -\bar{g}_{0i} \bar{u}_i - \bar{g}_{1i} \dot{\bar{u}}_i$ must be determined. The coefficients $\bar{g}_{0i}$ and $\bar{g}_{1i}$ are chosen to make each approximate closed-loop mode have a desired eigenvalue (the same eigenvalue that is desired of the distributed structure) or they can be obtained as the solution of a 2×2 matrix Ricatti equation for each approximate mode (the weights in the cost functional are the same as the weights that would be used for each actual mode). Finally, an approximate control force $f_m(x,t)$ is obtained in the form of Eqs. (12) and (13), where $G_0 = \mathrm{diag}(\bar{g}_{0i})$, $G_1 = \mathrm{diag}(\bar{g}_{1i})$ and $\bar{M}^{-1} = I$. Note that if pole placement is used, $\bar{g}_{0i} = g_{0i} + \omega_i^2 - \bar{\omega}_i^2$ and $\bar{g}_{1i} = g_{1i}$ $(i = 1,\ldots,n)$. Therefore, $\hat{g}_1 = O$ in Eq. (14b) and $\hat{g}_0 = O(|\omega_i^2 - \bar{\omega}_i^2|)$, where $\hat{g}_0$ is governed by the accuracy of the finite element model's eigenvalues. If optimal control is used,[15] it is easily shown that when the control weighting r_i is large, $\hat{g}_1 \approx O$ and $\hat{g}_0 = O(|\omega_i^2 - \bar{\omega}_i^2|)$, and when r_i is small, $O(\hat{g}_1) = O(\hat{g}_0) = O[\max\sqrt{1/r_i}\,|\omega_i - \bar{\omega}_i|]$.

An infinite number of implementations of f_m are theoretically permitted via the choice of π_c and π_s. Only implementations that use a finite number of discrete actuators and a finite number of discrete sensors are now considered. Let us denote by $\boldsymbol{\delta}_c(\boldsymbol{\delta}_s)$ an n_c-dimensional (n_s-dimensional) vector of functions corresponding to available discrete actuators (sensors), i.e., each entry in $\boldsymbol{\delta}_c(\boldsymbol{\delta}_s)$ is a function closely approximating a Dirac delta function which is nonzero only at x locating one actuator (sensor). Moreover, let us denote by $w_c(w_s)$ an n_c-dimensional (n_s-dimensional) vector of basis functions for a subspace of V denoted V_c^* (for V_s). The projections π_c and π_s now can be written as

$$\pi_c(\cdot) = \boldsymbol{\delta}_c^T (w_c, \boldsymbol{\delta}_c^T)^{-1} (w_c, \cdot) \tag{33a}$$

$$\pi_s(\cdot) = w_s^T (\boldsymbol{\delta}_s, w_s^T)^{-1} (\boldsymbol{\delta}_s, \cdot) \tag{33b}$$

Note that V_c is spanned by the entries of $\boldsymbol{\delta}_c$ and a subspace V_s^* of V' is spanned by the entries of $\boldsymbol{\delta}_s$. Therefore, π_c^* and π_s^* are projections onto V_c^* and V_s^*, respectively. Equations (33) permit many choices of w_c and w_s, i.e., of V_c^* and V_s. One possibility for w_c (w_s) is to use n_c (n_s) approximate eigenfunctions $\gamma^T \Phi_c$ ($\Phi^T \Phi_s$), where the n_c (n_s) columns of $\Phi_c(\Phi_s)$ are eigenvectors of the finite element model. Let us denote such a choice of w_c (w_s) by w_{mc} (w_{ms}). Another possibility for w_c (w_s) is to choose each entry as a function with local support, e.g., if $p = 1$ one could choose functions having the value one at one actuator (sensor) and varying linearly to zero at adjacent actuators (sensors). If $\bar{n}$ actuators were available it would be particularly convenient to choose $w_c = \gamma$. The choice of w_c (w_s) as piecewise functions with local support will be denoted by $w_{\ell c}$ ($w_{\ell s}$).

V. Numerical Examples

For simplicity, let us consider a uniform fixed-fixed beam in axial vibration. The length is taken to be unity, the mass per unit length is also unity, and the stiffness is 0.01. The natural damping is assumed to be extremely small and is ignored. The structure admits a simple closed-form eigensolution so that the independent modal space control method can be used to design a desired distributed controller. As an illustration, the

Table 1 Sensor distributions

No. of sensors	Vector of Dirac delta functions
6	$\delta_6^T = \{\delta(x-1/7), \delta(x-2/7), \delta(x-3/7), \delta(x-4/7), \delta(x-5/7), \delta(x-6/7)\}$
10	$\delta_{10}^T = \{\delta(x-3/21), \delta(x-4/21), \delta(x-6/21), \delta(x-8/21), \delta(x-9/21), \delta(x-12/21), \delta(x-13/21), \delta(x-15/21), \delta(x-17/21), \delta(x-18/21)\}$
13	$\delta_{13}^T = \{\delta(x-2/21), \delta(x-3/21), \delta(x-5/21), \delta(x-6/21), \delta(x-8/21), \delta(x-9/21), \delta(x-11/21), \delta(x-12/21), \delta(x-14/21), \delta(x-15/21), \delta(x-17/21), \delta(x-18/21), \delta(x-20/21)\}$
14	$\delta_{14}^T = \{\delta(x-2/21), \delta(x-3/21), \delta(x-5/21), \delta(x-6/21), \delta(x-7/21), \delta(x-9/21), \delta(x-10/21), \delta(x-11/21), \delta(x-12/21), \delta(x-14/21), \delta(x-15/21), \delta(x-16/21), \delta(x-18/21), \delta(x-19/21)\}$
18	$\delta_{18}^T = \{\delta(x-1/21), \ldots, \delta(x-6/21), \delta(x-8/21), \ldots, \delta(x-12/21), \delta(x-14/21), \ldots, \delta(x-20/21)\}$
20	$\delta_{20}^T = \{\delta(x-1/21), \delta(x-2/21), \ldots, \delta(x-20/21)\}$

Table 2 Comparison of eigenvalues for example 1

Mode No.	Dimension of the Finite Element Model 6	13	20	41	Test Controller	Dist. Controller
1	-.210163	-.210064	-.210047	-.210037	-.213710	-.213707
	+i .313914	+i .313851	+i .313839	+i .313832	+i .307269	+i .307192
2	-.195701	-.195372	-.195339	-.195323	-.220438	-.220432
	+i .640515	+i .640467	+i .640456	+i .640450	+i .627785	+i .627196
3	-.166835	-.165618	-.165553	-.165531	-.222117	-.222111
	+i .959226	+i .959112	+i .959101	+i .959094	+i .944105	+i .942124
4	-.133948	-.130159	-.130025	-.129991	-.222753	-.222747
	+i1.275896	+i1.275382	+i1.275360	+i1.275353	+i1.261178	+i1.256484
5	-.104267	-.094710	-.094467	-.094416	-.223057	-.223051
	+i1.593398	+i1.591634	+i1.591588	+i1.591577	+i1.579889	+i1.570717
6	-.083584	-.063279	-.062910	-.062841	-.223225	-.223219
	+i1.916177	+i1.910427	+i1.910334	+i1.910316	+i1.900768	+i1.884910
7	.000000	.000000	.000000	.000000	.000000	.000000
	+i2.224314	+i2.224314	+i2.224314	+i2.224314	+i2.224314	+i2.199115
8	-.088056	-.066233	-.065838	-.065764	.000000	.000000
	+i2.543617	+i2.547508	+i2.547565	+i2.547575	+i2.550921	+i2.513274
9	-.111599	-.101221	-.100958	-.100901	.000000	.000000
	+i2.878356	+i2.879193	+i2.879211	+i2.879214	+i2.881082	+i2.827433
10	-.146820	-.142608	-.142458	-.142420	.000000	.000000
	+i3.216366	+i3.216432	+i3.216432	+i3.216432	+i3.215248	+i3.141593
11	-.187572	-.186179	-.186104	-.186078	.000000	.000000
	+i3.563815	+i3.563697	+i3.563689	+i3.563686	+i3.553868	+i3.455752
12	-.225676	-.225287	-.225248	-.225229	.000000	.000000
	+i3.928863	+i3.928749	+i3.928737	+i3.928730	+i3.897388	+i3.769911
13	-.247443	-.247327	-.247308	-.247296	.000000	.000000
	+i4.345724	+i4.345599	+i4.345578	+i4.345565	+i4.246246	+i4.084071
14	.000000	.000000	.000000	.000000	.000000	.000000
	+i4.600870	+i4.600870	+i4.600870	+i4.600870	+i4.600870	+i4.398230
15	-.264821	-.264679	-.264655	-.264641	.000000	.000000
	+i4.860974	+i4.861090	+i4.861109	+i4.861120	+i4.961669	+i4.712389
16	-.252245	-.251802	-.251757	-.251735	.000000	.000000
	+i5.299616	+i5.299719	+i5.299729	+i5.299733	+i5.329031	+i5.026548
17	-.220088	-.218443	-.218354	-.218324	.000000	.000000
	+i5.698302	+i5.698400	+i5.698403	+i5.698404	+i5.703306	+i5.340708
18	-.180802	-.175594	-.175409	-.175362	.000000	.000000
	+i6.090558	+i6.090316	+i6.090306	+i6.090303	+i6.084800	+i5.654867
19	-.144117	-.130708	-.130368	-.130295	.000000	.000000
	+i6.484501	+i6.482790	+i6.482748	+i6.482739	+i6.473758	+i5.969026
20	-.118776	-.089493	-.088964	-.088864	.000000	.000000
	+i6.888209	+i6.880746	+i6.880630	+i6.880608	+i6.870342	+i6.283185

beam is also modeled by the finite element method and the independent modal space control method is used to design an approximate controller for the lowest six open-loop modes of vibration. Control gains are determined in modal space by the solution of a 2×2 steady-state matrix Ricatti equation as in Ref. 15 with $r_i=10$ ($i=1,2,\ldots,6$). Discrete force actuators and discrete displacement and velocity sensors are used to implement the various approximate controllers. Table 1 contains six vectors of Dirac delta functions (actually, very close approximations in V of Dirac delta functions) corresponding to the chosen sensor combinations. All examples use six actuators with $\delta_c=\delta_6$. Note that this choice of actuators

Table 3 Comparison of moduli for example 1

Mode No.	Dimension of the Finite Element Model 6	13	20	41	Test Controller	Dist. Controller
1	.377771	.377663	.377644	.377632	.374281	.374216
2	.669745	.669603	.669583	.669572	.665362	.664804
3	.973626	.973307	.973284	.973274	.969881	.967952
4	1.282908	1.282007	1.281972	1.281961	1.280699	1.276076
5	1.596806	1.594449	1.594389	1.594375	1.595557	1.586476
6	1.917999	1.911475	1.911370	1.911349	1.913830	1.898081
7	2.224314	2.224314	2.224314	2.224314	2.224314	2.199115
8	2.545141	2.548369	2.548415	2.548423	2.550921	2.513274
9	2.880518	2.880971	2.880980	2.880982	2.881082	2.827433
10	3.219715	3.219592	3.219586	3.219583	3.215248	3.141593
11	3.568748	3.568557	3.568545	3.568541	3.553868	3.455752
12	3.935339	3.935203	3.935188	3.935181	3.897388	3.769911
13	4.352763	4.352632	4.352609	4.352596	4.246246	4.084071
14	4.600870	4.600870	4.600870	4.600870	4.600870	4.398230
15	4.868183	4.868290	4.868308	4.868319	4.961669	4.712389
16	5.305615	5.305698	5.305705	5.305708	5.329031	5.026548
17	5.702551	5.702585	5.702585	5.702585	5.703306	5.340708
18	6.093241	6.092846	6.092831	6.092827	6.084800	5.654867
19	6.486103	6.484108	6.484059	6.484048	6.473758	5.969026
20	6.889233	6.881328	6.881205	6.881181	6.870342	6.283185

Table 4 Comparison of eigenvalues for example 2

Mode No.	Number of Sensors(Type of Projections) 6(L)	6(M)	13(L)	13(M)	20(L)	20(M)
1	-.210047	-.216728	-.210214	-.213066	-.210245	-.213424
	+i .313839	+i .311682	+i .310203	+i .307426	+i .309016	+i .307843
2	-.195339	-.221435	-.203058	-.216814	-.205512	-.218299
	+i .640456	+i .638786	+i .633510	+i .625111	+i .631083	+i .629054
3	-.165553	-.219560	-.183236	-.216438	-.189114	-.216896
	+i .959101	+i .962021	+i .951445	+i .944137	+i .948438	+i .946006
4	-.130025	-.215160	-.157137	-.212656	-.166614	-.213284
	+i1.275360	+i1.287683	+i1.268927	+i1.260044	+i1.265936	+i1.263654
5	-.094467	-.208714	-.128152	-.207662	-.140597	-.208252
	+i1.591588	+i1.620087	+i1.587098	+i1.579725	+i1.584495	+i1.582874
6	-.062910	-.197830	-.099132	-.201641	-.113257	-.202092
	+i1.910334	+i1.980778	+i1.907082	+i1.900660	+i1.904784	+i1.904189
7	.000000	.000000	.000000	.000000	.000000	.000000
	+i2.224314	+i2.224314	+i2.224314	+i2.224314	+i2.224314	+i2.224314
8	-.065838	-.218724	-.012079	.000000	.000000	.000000
	+i2.547565	+i2.488255	+i2.548993	+i2.550921	+i2.550921	+i2.550921
9	-.100958	-.227070	-.019144	.000000	.000000	.000000
	+i2.879211	+i2.860749	+i2.879647	+i2.881082	+i2.881082	+i2.881082
10	-.142458	-.237915	-.027788	.000000	.000000	.000000
	+i3.216432	+i3.213214	+i3.214078	+i3.215248	+i3.215248	+i3.215248
11	-.186104	-.248062	-.037736	.000000	.000000	.000000
	+i3.563689	+i3.568909	+i3.553986	+i3.553868	+i3.553868	+i3.553868
12	-.225248	-.255950	-.048130	.000000	.000000	.000000
	+i3.928737	+i3.937196	+i3.901512	+i3.897388	+i3.897388	+i3.897388
13	-.247308	-.255258	-.057150	.000000	.000000	.000000
	+i4.345578	+i4.353325	+i4.263062	+i4.246246	+i4.246246	+i4.246246
14	.000000	.000000	.000000	.000000	.000000	.000000
	+i4.600870	+i4.600870	+i4.600870	+i4.600870	+i4.600870	+i4.600870
15	-.264655	-.273298	-.069662	.074563	.000000	.000000
	+i4.861109	+i4.852912	+i4.934224	+i4.899085	+i4.961669	+i4.961669
16	-.251757	-.286198	-.075165	.256858	.000000	.000000
	+i5.299729	+i5.290114	+i5.316390	+i5.403048	+i5.329031	+i5.329031
17	-.218354	-.291614	-.074912	.093921	.000000	.000000
	+i5.698403	+i5.692646	+i5.698117	+i5.713030	+i5.703306	+i5.703306
18	-.175409	-.294079	-.070378	.184351	.000000	.000000
	+i6.090306	+i6.096748	+i6.084089	+i6.152492	+i6.084800	+i6.084800
19	-.130368	-.294575	-.062491	.099346	.000000	.000000
	+i6.482748	+i6.514114	+i6.475444	+i6.463196	+i6.473758	+i6.473758
20	-.088964	-.294234	-.050982	.116796	.000000	.000000
	+i6.880630	+i6.987269	+i6.880063	+i6.871106	+i6.870342	+i6.870342

renders modes 7, 14, etc. of the uncontrolled structure uncontrollable. Associated with each vector δ, one may define vectors of local support $w_{\ell 6}, w_{\ell 10}, \ldots$, as well as vectors $w_{m6}, w_{m10}, \ldots$ of global support as described in Sec. IV. The closed-loop eigenvalues obtained by applying the approximate controllers to a "test" finite element model with 42 equal length elements, i.e., 41 nodes, are compared in the following discussion. The comparisons are conveniently broken into three distinct representative examples. Many more illustrative examples are contained in Ref. 23.

First, example 1 compares four approximate controllers to 1) a desired control for the test model (a "test controller"), and 2) the desired distributed controller. The four approximate controllers are based on finite element models with 7, 14, 21, and 42 elements, respectively, i.e., 6, 13, 20, and 41 degrees of freedom. All four approximate controllers use six discrete sensors collocated with the six actuators, i.e., $\delta_s=\delta_c$ and $w_c=w_s=w_{\ell 6}$. This satisfies the criterion, $\pi_c=\pi_s^*$. One observes from Tables 2 and 3 that a higher accuracy structual

Table 5 Comprison of moduli for example 2

Mode No.	Number of Sensors(Type of Projections) 6(L)	6(M)	13(L)	13(M)	20(L)	20(M)
1	.377644	.379627	.374721	.374042	.373756	.374589
2	.669583	.676078	.665257	.661644	.663702	.665856
3	.973284	.986758	.968929	.968628	.967108	.970552
4	1.281972	1.305535	1.278620	1.277863	1.276854	1.281527
5	1.594389	1.633476	1.592264	1.593316	1.590721	1.596515
6	1.911370	1.990632	1.909657	1.911327	1.908148	1.914883
7	2.224314	2.224314	2.224314	2.224314	2.224314	2.224314
8	2.548415	2.497850	2.549022	2.550921	2.550921	2.550921
9	2.880980	2.869747	2.879710	2.881082	2.881082	2.881082
10	3.219586	3.222009	3.214198	3.215248	3.215248	3.215248
11	3.568545	3.577520	3.554186	3.553868	3.553868	3.553868
12	3.935188	3.945507	3.901809	3.897388	3.897388	3.897388
13	4.352609	4.360802	4.263446	4.246246	4.246246	4.246246
14	4.600870	4.600870	4.600870	4.600870	4.600870	4.600870
15	4.868308	4.860602	4.934715	4.899653	4.961669	4.961669
16	5.305705	5.297850	5.316921	5.409150	5.329031	5.329031
17	5.702585	5.700110	5.698609	5.713802	5.703306	5.703306
18	6.092831	6.103836	6.084496	6.155253	6.084800	6.084800
19	6.484059	6.520772	6.475745	6.463959	6.473758	6.473758
20	6.881205	6.993461	6.880252	6.872098	6.870342	6.870342

Table 6 Comparison of eigenvalues for example 3

Mode No.	Number of Sensors 6	10	14	18	20	Test Controller
1	-.213358	-.213418	-.213427	-.213427	-.213424	-.213710
	+i .312788	+i .310224	+i .308679	+i .308362	+i .307843	+i .307269
2	-.207938	-.214693	-.216316	-.217247	-.218299	-.220438
	+i .639745	+i .633008	+i .631294	+i .630234	+i .629054	+i .627785
3	-.190501	-.201587	-.211987	-.216896	-.216896	-.222117
	+i .960558	+i .955096	+i .949093	+i .946006	+i .946006	+i .944105
4	-.166947	-.192307	-.201360	-.208296	-.213284	-.222753
	+i1.280377	+i1.272846	+i1.268941	+i1.266000	+i1.263654	+i1.261178
5	-.140013	-.147945	-.202641	-.200433	-.208252	-.223057
	+i1.600930	+i1.599014	+i1.585459	+i1.585313	+i1.582874	+i1.579889
6	-.111617	-.140982	-.183102	-.202092	-.202092	-.223225
	+i1.926483	+i1.919568	+i1.911836	+i1.904189	+i1.904189	+i1.900768
7	.000000	.000000	.000000	.000000	.000000	.000000
	+i2.224314	+i2.224314	+i2.224314	+i2.224314	+i2.224314	+i2.224314
8	-.118700	-.074080	-.024619	.000000	.000000	.000000
	+i2.535747	+i2.544027	+i2.544049	+i2.550921	+i2.550921	+i2.550921
9	-.150674	-.116522	-.026552	.000000	.000000	.000000
	+i2.874088	+i2.881334	+i2.876959	+i2.881082	+i2.881082	+i2.881082
10	-.183627	-.097741	-.037254	-.030411	.000000	.000000
	+i3.215431	+i3.211236	+i3.212211	+i3.213211	+i3.215248	+i3.215248
11	-.214634	-.146824	-.023542	.000000	.000000	.000000
	+i3.565964	+i3.555866	+i3.556051	+i3.553868	+i3.553868	+i3.553868
12	-.240047	-.039956	-.098110	.000000	.000000	.000000
	+i3.932677	+i3.894353	+i3.894734	+i3.897388	+i3.897388	+i3.897388
13	-.251249	-.115147	-.055371	-.048776	.000000	.000000
	+i4.349355	+i4.250657	+i4.241279	+i4.246232	+i4.246246	+i4.246246
14	.000000	.000000	.000000	.000000	.000000	.000000
	+i4.600870	+i4.600870	+i4.600870	+i4.600870	+i4.600870	+i4.600870
15	-.268936	-.136848	-.027957	.000000	.000000	.000000
	+i4.857112	+i4.932995	+i4.961554	+i4.961669	+i4.961669	+i4.961669
16	-.268359	-.167185	.024771	-.068357	.000000	.000000
	+i5.295276	+i5.322314	+i5.322379	+i5.327379	+i5.329031	+i5.329031
17	-.252046	-.104783	-.084274	.000000	.000000	.000000
	+i5.696018	+i5.703960	+i5.699405	+i5.703306	+i5.703306	+i5.703306
18	-.226407	-.073864	-.087904	.000000	.000000	.000000
	+i6.092802	+i6.100990	+i6.099878	+i6.084800	+i6.084800	+i6.084800
19	-.194721	-.228797	.027582	-.079647	.000000	.000000
	+i6.492287	+i6.502080	+i6.475778	+i6.476324	+i6.473758	+i6.473758
20	-.159838	-.136762	-.039789	.000000	.000000	.000000
	+i6.902368	+i6.888069	+i6.868738	+i6.870342	+i6.870342	+i6.870342

Table 7 Comparison of moduli for example 3

Mode No.	Number of Sensors 6	10	14	18	20	Test Controller
1	.378627	.376545	.375279	.375017	.374589	.374281
2	.672690	.668425	.667327	.666627	.665856	.665362
3	.979266	.976138	.972479	.970552	.970552	.969881
4	1.291215	1.287291	1.284818	1.283021	1.281527	1.280699
5	1.607041	1.605844	1.598356	1.597933	1.596515	1.595557
6	1.929714	1.924738	1.920584	1.914883	1.914883	1.913830
7	2.224314	2.224314	2.224314	2.224314	2.224314	2.224314
8	2.538523	2.545105	2.544168	2.550921	2.550921	2.550921
9	2.878035	2.883689	2.877082	2.881082	2.881082	2.881082
10	3.220670	3.212723	3.212427	3.213354	3.215248	3.215248
11	3.572418	3.558896	3.556129	3.553868	3.553868	3.553868
12	3.939996	3.894558	3.895969	3.897388	3.897388	3.897388
13	4.356606	4.252216	4.241641	4.246512	4.246246	4.246246
14	4.600870	4.600870	4.600870	4.600870	4.600870	4.600870
15	4.864552	4.934893	4.961633	4.961669	4.961669	4.961669
16	5.302072	5.324939	5.322437	5.327817	5.329031	5.329031
17	5.701592	5.704923	5.700028	5.703306	5.703306	5.703306
18	6.097007	6.101438	6.100511	6.084800	6.084800	6.084800
19	6.495206	6.506104	6.475836	6.476814	6.473758	6.473758
20	6.904218	6.889426	6.868854	6.870342	6.870342	6.870342

model is a waste of effort when π_c and π_s are inaccurate. Table 2 also demonstrates that although the robust controller may cause substantial spillover, i.e., large real parts in the higher eigenvalues, it will be stabilizing. Note that modes 7 and 14 have zero real parts because they are uncontrollable.

Next, example 2 compares six approximate controllers which are based on a 21 element finite element model. As before, the control forces are implemented with six actuators. However, here 6, 13, or 20 sensors are used, i.e., $\delta_s = \delta_6$, δ_{13}, or δ_{20}. The vector w_c is taken as either $w_{\ell 6}$ or w_{m6}, where the choice is indicated in Tables 4 and 5 by (L) or (M), respectively. Likewise, $w_s = w_{\ell 6}$, $w_{\ell 13}$, w_{m13}, $w_{\ell 20}$ or w_{m20} as indicated by 6(L), 6(M), 13(L), 13(M), 20(L), or 20(M), respectively. Tables 4 and 5 compare projections based on functions of local support with those based on functions of global support. They also demonstrate the effects of increasing the number of sensors. For example, columns 6(M) and 13(M) show that improving the sensing, without satisfying the robustness criterion, may actually cause instability.

Finally, example 3 compares five approximate controllers based on a 21 element finite element model and the test controller. Again, six actuators are used, where each approximate controller uses $w_c = w_{m6}$. The difference is that 6, 10, 14, 18, or 20 sensors are used with $\delta_s = \delta_6$, δ_{10}, δ_{14}, δ_{18}, or δ_{20}, respectively, as well as $w_s = w_{\ell 6}$, $w_{\ell 10}$, $w_{\ell 14}$, $w_{\ell 18}$, or $w_{\ell 20}$, respectively. Tables 6 and 7 show the effect of implementing an approximate controller with an increasing number of sensors. Although all eigenvalues converge in modulus, the real parts of the 16th and 19th eigenvalues alternate signs.

VI. Conclusions

We distinguish among designing control forces for a reduced-order model, implementing the control forces with a number of actuators, and estimating the distributed state from a number of sensors. By introducing three corresponding projection operations, the effects on the actual closed-loop eigenvalues of using a reduced-order controller are studied. The significance of modal space control and independent modal space control arises naturally. Moreover, the controller is shown to be robust when the control force projection operator is the adjoint of the observations projection operator.

The discussion of this paper is concerned with a force projection operator and an observations projection operator that are not time varying. Most of the discussion can be easily extended to the more practical situation in which the observations projection is realized by an observer of some kind and/or the force projection is realized by a dual observer.[24] The use of an observer and/or a dual observer permits accurate controllers to be obtained with only a few sensors and actuators. This paper does not consider an observer and a dual observer for simplicity.

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